Matrix approach to rough sets through representable matroids over a field

Aiping Huang, William Zhu *

Lab of Granular Computing,
Zhangzhou Normal University, Zhangzhou 363000, China

Abstract. Rough sets was proposed to deal with the vagueness and incompleteness of knowledge in information systems. Representable matroid over field is an important branch of matroid theory. In this paper, we use one matrix approach, namely, null space, to study rough sets through representable matroids over a field. First, we introduce an approach to obtain a matroid from an equivalence relation and we find that it is representable over a field. Second, we use the null space of a corresponding representation for a representable matroid to study rough sets over a filed. It is interesting to find that the set of circuits of the matroid has closely relation to the null space of a corresponding representation for the matroid over a field. Third, we study how to induce an equivalence relation from the null space of a matrix over binary field. We find that there is an one-to-one corresponding between the equivalence relations and the minimal null spaces of the matrices defined in the paper. In a word, this work indicates that we can study rough sets from the viewpoint of matrix.

Keywords. Rough sets, Representable matroid, Representation, Null space.

Introduction

The vagueness and incompleteness of knowledge are commom phenomenons in information systems. Rough set theory [22], based on equivalence relations, was proposed by Pawlak in hybrid approaches to improve the performance of data analysis tools. This technique has led to many practical applications in various areas such as attribute reduction [3,10,13,20,28], feature selection[4,7,11,23], rule extraction[1,6,8,24] and so on. Matroid theory has been promoted further to study rough set theory and its applications [9].

Matroids [14,21] develops mainly out of a deep examination of the properties of independence and dimension in vector spaces. There are two ways to present the matroids, namely, vector matroid and representable matroid. In fact, a representable matroid is equivalence to a vector matroid, although it may be presented differently. A basic problem in matroid theory is to characterize the matroids that may be represented over field. It determines whether we can study rough sets by a new viewpoint, namely, matrix one, or not. In recent years, there are many fruitful achievements about the connection between these two theories [2,15,25,26,29].

Matrix as an approach to study rough sets exists in many papers [12,16,17,18]. It also has numerous applications both in mathematics and other sciences, for example,

* Corresponding author. E-mail: williamfengzhu@gmail.com (William Zhu)

text mining and automated thesaurus compilation makes use of document-term matrices such as term frequency inverse document frequency to track frequencies of certain words in several documents. In addition to this, null space is one of the applications of matroid which exists in many areas such as numerical analysis, matrix decomposition and so on.

In this paper, we use null space approach to study rough sets through representable matroids over a field, especially, binary field. First, an approach to induce a matroid is introduced from the viewpoint of circuit and we find that the matroid is representable over a filed. Second, by the representable matroid, we use one matrix approach, namely, null space, to study rough sets over a filed. It is interesting to find that the set of circuits of the representable matroid is equal to the minimal null space of a representation for the matroid over a field. Especially, over binary field, many interesting results of the matroid induced by an equivalence relation are established by the null space of the representation for the matroid. Third, we study that how to induce an equivalence relation from the null space of a matrix, and we find that there is an one-to-one correspondence between the equivalence relations and the minimal null spaces of the matrices defined in the paper. Moreover, there also has an one-to-one correspondence between 2-circuit matroids and the minimal null spaces of the matrices. In a word, this work indicates one interesting view, namely, matrix one, to study rough sets through representable matroids over a field.

The rest of this paper is arranged as follows. Section 1 reviews some fundamental concepts related to rough sets, matroids and linear algebra. In section 2, we study the matroidal structure to rough sets through matrix over a field. Section 3 uses matrix null space approach to study rough sets through a representation for a representable matroid over a field. In section 4, we construct a rough set structure from the matrix null space over binary field. In section 5, we conclude this paper.

1 Basic definitions

In this section, we present some fundamental concepts about rough sets, matroids and linear algebra. First of all, we review the basic concepts of rough sets.

1.1 Rough sets

In Pawlak's rough set theory, the lower and upper approximation operations are two key concepts. An equivalence relation, that is, a partition, is the simplest formulation of the lower and upper approximation operations. So we first introduce the concept of partition.

Definition 1. *(Partition) Let U be a universe of discourse, $\mathcal{P}$ a family of non-empty subsets of U. If any two subsets in $\mathcal{P}$ are disjoint and $\bigcup\mathcal{P} = U$, then $\mathcal{P}$ is called a partition of U.*

Let U be a finite set and R be an equivalence relation on U. R will generate a partition $U/R = \{P_1, P_2, \cdots, P_s\}$ on U, where $P_1, P_2, \cdots, P_s$ are the equivalence classes generated by R. $\forall X \subseteq U$, the lower and upper approximations of X, are,

respectively, defined as follows:
$R_*(X) = \bigcup\{P_i \in U/R : P_i \subseteq X\}$,
$R^*(X) = \bigcup\{P_i \in U/R : P_i \bigcap X \neq \emptyset\}$.

1.2 Linear algebra

In this subsection, we introduce some basic concepts of linear algebras which will be used in this paper. First, we introduce the concept of a filed.

Definition 2. *(Field)[27] A field is defined as a set together with two operations, usually called addition and multiplication, and denoted by $+$ and $\cdot$, respectively, such that the following axioms hold (subtraction and division are defined implicitly in terms of the inverse operations of addition and multiplication):*
(1): For all $a, b \in F$, $a + b \in F$ and $a \cdot b \in F$.
(2): For all $a, b, c \in F$, $a + (b + c) = (a + b) + c$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
(3): For all $a, b \in F$, $a + b = b + a$ and $a \cdot b = b \cdot a$.
(4): There exists an element of F, called the additive identity element and denoted by 0, such that for all $a \in F$, $a + 0 = a$. Likewise, there is an element, called the multiplicative identity element and denoted by 1, such that for all $a \in F$, $a \cdot 1 = a$.
(5): For every $a \in F$, there exists an element $-a \in F$ such that $a+(-a) = 0$. Similarly, for any $a \in F$ other than 0, there exists an element $a^{-1} \in F$ such that $a \cdot a^{-1} = 1$.
(6) For all $a, b, c \in F$, the following equalities holds: $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$.
A field is therefore an algebraic structure $< F, +, \cdot, -, ^{-1}, 0, 1 >$.

Generally, for a field F and positive integer n, $V(n, F)$ denotes the $n-$dimensional vector space over F. Any element of $V(n, F)$ is denoted as $\mathbf{v} = (v_1, v_2, \cdots, v_n)^T$ where $v_i \in F$ ($1 \leq i \leq n$). The operations on $V(n, F)$ are established as follows. For all $\mathbf{v} = (v_1, v_2, \cdots, v_n)^T \in V(n, F)$ and $\mathbf{v}^{'} = (v_1^{'}, v_2^{'}, \cdots, v_n^{'})^T \in V(n, F)$ and $k \in F$, $k\mathbf{v} = (kv_1, kv_2, \cdots, kv_n)^T$ and $\mathbf{v} + \mathbf{v}^{'} = (v_1 + v_1^{'}, v_2 + v_2^{'}, \cdots, v_n + v_n^{'})^T$.

Definition 3. *[5] Let F be a filed and A an $m \times n$ matrix over F. The null space of an $m \times n$ matrix A, written as $\mathcal{N}(A)$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notion, $\mathcal{N}(A) = \{\mathbf{x} \in V(n, F) : A\mathbf{x} = \mathbf{0}\}$.*

$\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n \in V(n, F)$ are said to be linear independent over F if there exists $x_1, x_2, \cdots, x_n \in F$ such that the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$ has only the trivial solution. $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is said to be linearly dependent over F if there exist $c_1, c_2, \cdots, c_p \in F$, not all zero, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$. The rank of matrix A over F is the maximum number of linearly independent columns in A and the maximum number of linearly independent columns in A^T (rows in A), and we denote it as $r_F(A)$.

1.3 Matroids

Matroid theory borrows extensively from the terminology of linear algebra and graph theory, largely because it is the abstraction of various notions of central importance in these fields, such as independent set, base, rank function.

Definition 4. *(Matroid) [14,21] A matroid is an ordered pair* $(U, \mathcal{I})$ *consisting of a finite set* U *and a collection* $\mathcal{I}$ *of subsets of* U *satisfying the following three conditions:*
(I1) $\emptyset \in \mathcal{I}$*;*
(I2) If $I \in \mathcal{I}$ *and* $I^{'} \subseteq I$*, then* $I^{'} \in \mathcal{I}$*;*
(I3) If $I_1, I_2 \in \mathcal{I}$ *and* $|I_1| < |I_2|$*, then there is an element* $e \in I_2 - I_1$ *such that* $I_1 \bigcup e \in \mathcal{I}$*, where* $|X|$ *denotes the cardinality of* X*.*

Let $M(E, \mathcal{I})$ be a matroid. The members of $\mathcal{I}$ are the independent sets of M. A set in $\mathcal{I}$ is maximal, in the sense of inclusion, is called a base of the matroid M. If $A \notin \mathcal{I}$, A is called dependent set. In the sense of inclusion, a minimal dependent subset of U is called a circuit of the matroid M. The rank function of a matroid is a function $r_M : 2^E \to N$ defined by $r_M(X) = max\{|I| : I \subseteq X, I \in \mathcal{I}\}$ $(X \subseteq E)$.

Definition 5. *(Circuit axiom)[14,21] Let* $\mathcal{C}$ *be a family of* U*. There exists a matroid* M *such that* $\mathcal{C} = \mathcal{C}(M)$ *if and only if* $\mathcal{C}$ *satisfies the following conditions:*
(C1) $\emptyset \in \mathcal{C}$*;*
(C2) $C_1, C_2 \in \mathcal{C}$*, if* $C_1 \subseteq C_2$*, then* $C_1 = C_2$*;*
(C3) $C_1, C_2 \in \mathcal{C}$*, if* $C_1 \neq C_2$ *and* $x \in C_1 \bigcap C_2$*, then there exists* $C_3 \in \mathcal{C}$ *such that* $C_3 \subseteq C_1 \bigcup C_2 - \{x\}$*.*

The following type of matroidal structure is a generalization of linear algebra.

Definition 6. *(Vector matroid)[14,21] Let* U *be the set of column labels of an* $m \times n$ *matrix* A *over a field* F*, and let* $\mathcal{I}$ *be the set of subsets* X *of* U *for which the columns labeled by* X *is linearly independent in the vector space* $V(m, F)$*. Then* $(E, \mathcal{I})$ *is a matroid. It called the vector matroid of* A*, which denoted by* $M_F[A]$*.*

Example 1. Let A be a matrix over the field $\mathbf{R}$.

$$A = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Then $U = \{1, 2, 3, 4, 5\}$ and $\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}\}$ and $\mathcal{C} = \{\{1, 2\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}$.

Representable matroid is equivalence to a vector matroid, although it may be presented differently. The definition is presented as follows.

Definition 7. *(Representable matroid)[14,21] Let* M *be a matroid and* F *be a field.* M *is said to be representable over* F *or* $F-$ *representable, if there exists a field* F *and a matrix* A *over* F *such that* $M = M_F[A]$*.* A *is called a representation for* M *over* F *or an* $F-$*represention for* M*.*

It is natural for us to connect two matroidal structures with each other, and the isomorphism is an efficient tool to solve this problem.

Definition 8. *(Isomorphism)[14,21] Let* $M_1 = M(U_1, \mathcal{I}_1)$ *and* $M_2 = M(U_2, \mathcal{I}_2)$ *be two matroids.* M_1 *and* M_2 *are isomorphic, denoted as* $M_1 \cong M_2$*, if there is a bijection* $\varphi : U_1 \to U_2$ *such that* $I \in \mathcal{I}_1$ *if and only if* $\varphi(I) \in \mathcal{I}_2$*.*

For convenience, we introduce some symbols.

Definition 9. *[14,21] Let $\mathcal{A}$ be a family of U. One can denote*
$Low(\mathcal{A}) = \{X \subseteq U : \exists A \in \mathcal{A}\ such\ that\ A \subseteq X\}$;
$Min(\mathcal{A}) = \{X \subseteq U : \forall A \in \mathcal{A}, if\ Y \subseteq X, then\ X = Y\}$.

2 Matroidal structure of rough sets through matrix over a field

In this section, we study the matroidal structure of rough sets though matrix approach over a filed. First of all, an approach to induce a matroid by an equivalence relation is provided.

Proposition 1. *[26] Let R be an equivalence relation on U and $U/R = \{P_1, P_2, \cdots, P_s\}$.*

$$\mathcal{C}(R) = \{\{x, y\} \subseteq U | \{x, y\} \subseteq P_i, \forall i \in \{1, 2, \cdots, s\}\}$$

satisfies circuit axiom $(C1)$, $(C2)$ and $(C3)$. There exists a matroid M such that $\mathcal{C}(M) = \mathcal{C}(R)$, and we denote this matroid as $M(R)$.

Wang et al. also define a special type of matroid, namely, 2-circuit matroid as follows.

Proposition 2. *(2-circuit matroid)[26] Let $M = (U, \mathcal{I})$ be a matroid, M is called a $2-$circuit matroid if $|C| = 2$ for all $C \in \mathcal{C}(M)$.*

They point out that the equivalence relations and the $2-$circuit matroids are algebraic system isomorphic in [26]. That provides a sound theoretical foundation for equivalently investigating rough sets with the matroids. Proposition 1 proposes an approach to induce a matroid from the viewpoint of set theory. Over a field, can we obtain a matroidal structure of rough sets through a matrix? To solve this problem, we define a matrix from an equivalence relation firstly.

Definition 10. *[12] Let R be an equivalence relation on $U = \{x_1, x_2, \cdots, x_n\}$ and $U/R = \{P_1, P_2, \cdots, P_s\}$. We denote a matrix $B(R) = (b_{ij})_{s\times n}$ as follows:*

$$b_{ij} = \begin{cases} 1 & x_j \in P_i, \\ 0 & x_j \notin P_i. \end{cases} \tag{1}$$

$B(R)$ is called a matrix representation of R.

Remark 1. Any column of $B(R)$ has only one non-zero element and it dose not contain zero row and zero column vectors.

For a filed F, we denote 1 and 0 as the multiplicative and additive identity elements of the field F, respectively. Thus $B(R)$ is the matrix over F. Interchanging two columns of $B(R)$ but does not interchange the labels of them, we obtain a new matrix $B^{'}(R)$. The $M_F[B(R)]$ and $M_F[B^{'}(R)]$ are isomorphic, which is an important result in matroid theory. Moreover, if we interchange the two columns of $B(R)$ and also interchange the labels of the them, then we obtain a new matrix $B^{''}(R)$. The $M_F[B(R)]$ and $M_F[B^{''}(R)]$ generate the same matroid. In other word, a F-representable matroid may not have unique F-representation.

Example 2. Suppose R is an equivalence relation on $U = \{1,2,3,4,5\}$ and $U/R = \{\{1,3\},\{2,4,5\}\}$ and $B(R)$ is a matrix representation of R and M the vector matroid of $B(R)$ over F. Then

$$B(R) = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix} \end{array}.$$

Interchanging the $2th$ column and the $3th$ column, we obtain

$$B^{'}(R) = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Interchanging the $2th$ column and the $3th$ column and the labels of them are also interchanged, we obtain

$$B^{''}(R) = \begin{array}{c} \begin{matrix} 1 & 3 & 2 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Hence, we have $M_F[B(R)] \cong M_F[B^{'}(R)]$ and $M_F[B(R)] = M_F[B^{''}(R)]$, that is, $B(R)$ and $B^{''}(R)$ are $F-$representation for M

What is the relationship between the matrix representation of a equivalence relation and the matroid induced by equivalence relation? Dose the matroid is the vector matroid induced by the matrix? As we know, the vector matroid is defined by the linear independence of the columns of a matrix in vector space and the linear independence of columns of a matrix has closely relation to field. The field is different, the vector matroid may different. The following proposition studies the relation between $B(R)$ and $M(R)$ over a field F from the viewpoint of circuit firstly.

Proposition 3. *Let R be an equivalence relation on U and $B(R)$ a matrix representation of R over F. $\mathcal{C}(M(R)) = \mathcal{C}(M_F[B(R)])$.*

Proof. Let $U = \{x_1, x_2, \cdots, x_n\}$ and $B(R) = [\beta_1, \beta_2, \cdots, \beta_n]$ be labeled, in order, by $x_1, x_2, \cdots, x_n$. In order to prove $\mathcal{C}(M(R)) = \mathcal{C}(M_F[B(R)])$, we need to prove only $\mathcal{C}(M(R)) = Min\{X \subseteq U$: The columns of $B(R)$ labeled by X are linearly dependent in $V(n,F)\}$. Suppose $U/R = \{P_1, P_2, \cdots, P_s\}$. $\forall \{x_i, x_j\} \in \mathcal{C}(M(R))$, there exists $P_k \in U/R$ such that $\{x_i, x_j\} \subseteq P_k$. Base on the definition of $B(R)$, we know that $\beta_i = \beta_j$. Thus β_i and β_j are linearly dependent in $V(n,F)$, that is, $\mathcal{C}(M(R)) \subseteq \{X \subseteq U$: The columns of $B(R)$ labeled by X are linearly dependent in $V(n,F)\}$. For all $X \in \{ X \subseteq U$: The columns of $B(R)$ labeled by X are linearly dependent in $V(n,F)$ $\}$, then X is an dependent set in $M(R)$; otherwise, X dose not contain circuits, that is, the columns of $B(R)$ labeled by X are different and they has a identity matrix as their submatrix. Hence the columns of $B(R)$ labeled by X are linearly independent in $V(n,F)$. That implies contradictory. Hence, $\mathcal{C}(M(R)) = Min\{X \subseteq U$: The columns of $B(R)$ labeled by X are linearly dependent in $V(n,F)\}$, that is, $\mathcal{C}(M(R)) = \mathcal{C}(M_F[B(R)])$.

As we know, a circuit decides only one matroid, then we can obtain the following theorem. In fact, the theorem indicates that $M(R)$ is F-representable matroid, and $B(R)$ is a F-representation for $M(R)$.

Theorem 1. *Let R be an equivalence relation on U. $M(R) = M_F[B(R)]$.*

Proof. Base on the circuit axiom and Proposition 3, we obtain the result.

As we know, the common filed is finite field or Galois field which is a field that contains a finite number of elements. The following definition introduces the concept of binary field.

Definition 11. *[27] Let $GF = \{0, 1\}$. If the operations of GF is defined in Table 1, then $(GF, +, \cdot)$ is called binary field and we denote it as $GF(2)$.*

Table 1. Operations of GF

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

$GF(2)$ is a special field, thus it is not difficult for us to obtain the following corollary.

Corollary 1. *Let R be an equivalence relation on U. $M(R) = M_{GF(2)}[B(R)]$.*

The matroid induced by an equivalence relation by the approach in the paper is a vector matroid induced by $B(R)$ over a field because of the particularity of the equivalence relation. However, in many cases, the vector matroids induced by the same matrix over different field may not be the same. The example below illustrates this viewpoint.

Example 3. Suppose A is a matrix as follows:

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

We may as well suppose $A = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. Because $det([\alpha_4, \alpha_5, \alpha_6]) = -2 \neq 0$ over real number field but $det([\alpha_4, \alpha_5, \alpha_6]) = 0$ over $GF(2)$, then $\alpha_4, \alpha_5, \alpha_6$ over real number field are linearly independent, and they are linearly dependent over $GF(2)$.

3 Matrix null space approach to rough sets over a field

In this section, we use a matrix approach, namely, null space, to study rough sets. A F-representable matroid is defined by the linear relativity, over the F, of the columns of a F-representation for the matroid, while null space has closely relation to linear relativity of vectors. This inspire us to introduce null space to study rough sets over a field. We find that the circuits of a F-representable matroid has closely relation to the null space of a F-representation for the matroid. Inspired by the null space, we also introduce the other space which has closely relationship to the base of a matroid to study rough sets. First of all, we propose an operator to connect vectors with classical sets.

Definition 12. *Let $U = \{x_1, x_2, \cdots, x_n\}$. We define a mapping $\theta : V(n, F) \to 2^U$ as follows: for all $\mathbf{v} = (v_1, v_2, \cdots, v_n)^T \in V(n, F)$, $\theta(\mathbf{v}) = \{x_i \in U : v_i \neq 0, 1 \leq i \leq n\}$.*

As we know, a F-representable matroid is defined by the collection of subset X of U such that the columns of a F-representation for the matroid labeled by X are linearly independent in $V(n, F)$. According to the characteristic of the null space, it is natural for us to connect the dependent set of a F-representable matroid with the null space of a F-representation for the matroid.

Proposition 4. *Let A be an $m \times n$ matrix over F and M be a matroid. If $M = M_F[A]$, then $\{\theta(\mathcal{N}(A)) - \emptyset\} \subseteq \mathcal{D}(M)$.*

Proof. We may as well suppose $A = [I_r|D] = [\alpha_1, \alpha_2, \cdots, \alpha_n]$, where $r = r(M)$. And the columns of A be labeled, in order, by $a_1, a_2, \cdots, a_n$. If $r = n$, then $A = I_n$, where I_n is the $n \times n$ identity matrix. Thus $D(M) = \emptyset$ and $\{\theta(\mathbf{x}) - \emptyset : A\mathbf{x} = \mathbf{0}\} = \emptyset$. Hence we obtain the result. If $r < n$, then $\{\theta(\mathbf{x}) - \emptyset : A\mathbf{x} = \mathbf{0}\} \neq \emptyset$. For all $D \in \theta(\mathcal{N}(A)) - \emptyset$, there exists $\mathbf{x} \in V(n, F) - \mathbf{0}$ such that $A\mathbf{x} = \mathbf{0}$ and $\theta(\mathbf{x}) = D$. Suppose $x_{i_1}, x_{i_2}, \cdots, x_{i_s} (s \geq 1)$ are non-zero components of $\mathbf{x}$, respectively. Then $D = \theta(\mathbf{x}) = \{a_{i_1}, a_{i_2}, \cdots, a_{i_s}\}$. Thus $\mathbf{0} = A\mathbf{x} = \sum_{i=1}^{n} x_i\alpha_i = \sum_{j=1}^{s} x_{i_j}\alpha_{i_j}$, that is, the column vectors labeled by D are linearly dependent over F. Hence, $\theta(\mathcal{N}(A)) - \emptyset \subseteq \mathcal{D}(M)$.

Conversely, the dependent set of $M_F[A]$ may not be in $\theta(\mathcal{N}(A)) - \emptyset$. The following example illustrates that viewpoint.

Example 4. Let us revisit Example 2. We may as well suppose $B(R) = [\beta_1, \beta_2, \beta_3, \beta_4, \beta_5]$ which the columns are labeled, in order, by $1, 2, 3, 4, 5$. It is easy to check that $\beta_1, \beta_2, \beta_4$ are linearly dependent in $V(n, GF(2))$, that is, $\{1, 2, 4\} \in \mathcal{D}(M_{GF(2)}[B(R)])$. Suppose $\mathbf{x} = (1, 1, 0, 1, 0)^T$. We know that $\{1, 2, 4\} = \theta(\mathbf{x})$, but $B(R)\mathbf{x} = (1, 0)^T \neq \mathbf{0}$. Thus $\{1, 2, 4\} \notin \theta(\mathcal{N}(A)) - \emptyset$.

We try to reduce the elements of $\theta(\mathcal{N}(A)) - \emptyset$, and it is interesting to find that the set of circuits of a F- representable matroid is just the minimal null space of a F-representation for the matroid.

Theorem 2. *Let A be an $m \times n$ matrix over F and M be a matroid. If $M = M_F[A]$, then $\mathcal{C}(M) = Min(\{\theta(\mathcal{N}(A)) - \emptyset\})$.*

Proof. Suppose $A = [I_r|D] = [\alpha_1, \alpha_2, \cdots, \alpha_n]$ be labeled, in order, by $a_1, a_2, \cdots, a_n$ where $r = r_M$. If $r = n$, then $\{\theta(\mathbf{x}) - \emptyset : A\mathbf{x} = \mathbf{0}\} = \emptyset$ and $\mathcal{C}(M) = \emptyset$. Thus we obtain the result. We may as well suppose $r < n$ and $C = \{a_{i_1}, a_{i_2}, \cdots, a_{i_s}\}$ is a circuit of M, where $s \leq n$. Then for some elements $k_{i_1}, k_{i_2}, \cdots, k_{i_s}$ of F which all are not equal to zero, we have $\mathbf{0} = k_{i_1}\alpha_{i_1} + k_{i_2}\alpha_{i_2} + \cdots + k_{i_s}\alpha_{i_s} = \sum_{j=1}^{s} k_{i_j}\alpha_{i_j} + \sum_{t \neq i_j, 1 \leq j \leq s} 0 \cdot \alpha_t = A\mathbf{x}$, where $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$, and $x_p = k_{i_j}$ if and only if $p = i_j, 1 \leq j \leq s$. Since $C \in \mathcal{C}(M)$, $k_{i_j} \neq 0$ for all $j \in \{1, 2, \cdots, s\}$; otherwise, $\theta(\mathbf{x}) \subseteq C$ and $\theta(\mathbf{x})$ is a dependent set of M for the columns of A labeled by $\theta(\mathbf{x})$ are linearly dependent in $V(n, F)$. Then there exists $C_1 \in \mathcal{C}(M)$ such that $C_1 \subseteq \theta(\mathbf{x}) \subseteq C$. According to the circuit axiom, we can obtain the contradictory. Hence, $\theta(\mathbf{x}) = C$, that is, $C \in \{\theta(\mathcal{N}(A)) - \emptyset\}$. Thus $\mathcal{C}(M) \subseteq \{\theta(\mathcal{N}(A)) - \emptyset\}$. In order to complete the proof, we need to prove that for all $D \in \{\theta(\mathcal{N}(A)) - \emptyset\}$, we have $D \in \mathcal{D}(M)$. According to Proposition 4, we obtain the result.

As we know, a F-representable matroid may not be has only one F-representation. However, Theorem 2 indicates that the minimal null spaces of these F-representations are unique. Before to present the accurately expression of the null space of a F-representation for a F-representable matroid, we introduce an important result of linear algebra first.

Lemma 1. *[27] Let A be an $m \times n$ matrix over F and $V(n, F)$ a $n-$dimensional vector space over F. $A(k_1\beta + k_2\gamma) = k_1 A\beta + k_2 A\gamma$ for all $k_1, k_2 \in F$ and $\beta, \gamma \in V(n, F)$.*

Proposition 5. *Let A be an $m \times n$ matrix over F and M be a matroid. If $M = M_F[A]$, then $\{\theta(\mathcal{N}(A)) - \emptyset\} = \{X \subseteq U : X$ is a disjoint union of some elements of $\mathcal{C}(M)\}$.*

Proof. For all $X \in \{X \subseteq U : X$ is a disjoint union of some elements of $\mathcal{C}(M)\}$, we may as well suppose $C_1, C_2, \cdots, C_t \in \mathcal{C}(M)$ are disjoint circuits such that $X = \bigcup_{i=1}^{t} C_i$. Assume $\mathbf{x}_i \in V(n, F)$ $(1 \leq i \leq t)$ such that $C_i = \theta(\mathbf{x}_i)$. According to Theorem 2, we know $A\mathbf{x}_i = \mathbf{0}$. Thus we have $X = \theta(\sum_{i=1}^{t} \mathbf{x}_i)$ and $A(\sum_{i=1}^{t} \mathbf{x}_i) = \sum_{i=1}^{t} A\mathbf{x}_i = \mathbf{0}$. Hence, $X \in \{\theta(\mathbf{x}) - \emptyset : A\mathbf{x} = \mathbf{0}\}$, that is, $\{X \subseteq U : X$ is a disjoint union of some elements of $\mathcal{C}(M)\} \subseteq \{\theta(\mathcal{N}(A)) - \emptyset\}$. For all $X \in \{\theta(\mathcal{N}(A)) - \emptyset\}$, there exists $\mathbf{x} \neq \mathbf{0}$ such that $X = \theta(\mathbf{x})$ and $A\mathbf{x} = \mathbf{0}$. According to Proposition 4, we know X is a dependent set of M. Then there exists $C_1 \in \mathcal{C}(M)$ such that $C_1 \subseteq X$. We may as well suppose $\mathbf{x}_1 \in V(n, F)$ such that $\theta(\mathbf{x}_1) = C_1$. If $X - C_1 = \emptyset$, then we complete the proof. If $X - C_1 \neq \emptyset$, then $X - C_1$ is a dependent set of M because $A(\mathbf{x} - \mathbf{x}_1) = A\mathbf{x} - A\mathbf{x}_1 = \mathbf{0} - \mathbf{0} = \mathbf{0}$. Similarly, there exists $C_2 \in \mathcal{C}(M)$ such that $C_2 \subseteq X - C_1$. Suppose $\mathbf{x}_2 \in V(n, F)$ such that $C_2 = \theta(\mathbf{x}_2)$. If $X - C_1 - C_2 = \emptyset$, then we complete the proof. If $X - C_1 - C_2 \neq \emptyset$, then $X - C_1 - C_2$ is a dependent set of M because $A(\mathbf{x} - \mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x} - A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{0} - \mathbf{0} - \mathbf{0} = \mathbf{0}$. Since X is finite, we can find $C_1, C_2, \cdots, C_t$ using the above method such that $X = \bigcup_{i=1}^{t} C_i$. It is clear that $C_1, C_2, \cdots, C_t$ are disjoint sets of $\mathcal{C}(M)$. Hence, $X \in \{X \subseteq U : X$ is a disjoint union of some elements of $\mathcal{C}(M)\}$, that is, $\theta(\mathcal{N}(A)) - \emptyset \subseteq \{X \subseteq U : X$ is a disjoint union of some elements of $\mathcal{C}(M)\}$. Hence we obtain the result.

It is natural for us to obtain the following three corollaries based on Proposition 4, Theorem 2 and Proposition 5.

Corollary 2. *Let R be an equivalence relation on U and $B(R)$ a matrix representation of R over $GF(2)$. $\theta(\mathcal{N}(B(R))) - \emptyset \subseteq \mathcal{D}(M(R))$.*

Corollary 3. *Let R be an equivalence relation on U and $B(R)$ a matrix representation of R over $GF(2)$. $\mathcal{C}(M(R)) = Min(\{\theta(\mathcal{N}(B(R))) - \emptyset\})$.*

Corollary 4. *Let R be an equivalence relation on U and $B(R)$ a matrix representation of R over $GF(2)$. $\{\theta(\mathcal{N}(B(R))) - \emptyset\} = \{X \subseteq U : X$ is a disjoint union of some elements of $\mathcal{C}(M(R))\}$.*

Inspired by the null space, we introduce the other space, namely, $\{\theta(\mathbf{v}) : A\mathbf{v} = \mathbf{1}\}$, to study rough sets. According to the characteristic of $B(R)$, we know the space $B(R)\mathbf{v} = \mathbf{1}$ over $GF(2)$ has solutions, and we find the space has closely relation to the base of a matroid.

Theorem 3. *Let R be an equivalence relation on U and $B(R)$ a matrix representation of R over $GF(2)$. $\mathcal{B}(M(R)) = Min(\{\theta(\mathbf{v}) : B(R)\mathbf{v} = \mathbf{1}\})$.*

Proof. Let $U = \{x_1, x_2, \cdots, x_n\}$ and $U/R = \{P_1, P_2, \cdots, P_s\}$ and $B(R) = [\beta_1, \beta_2, \cdots, \beta_n]$ which the columns are labeled, in order, by $x_1, x_2, \cdots, x_n$. We know that $s \leq n$. For all $B \in \mathcal{B}(M(R))$, since $M(R) = M_{GF(2)}([B(R)])$, we have $B \in \mathcal{B}(M_{GF(2)}([B(R)])$. According to the definition of matrix $B(R)$ and the fact that R is an equivalence relation, then $r(B(R)) = s$. We may as well suppose $B = \{x_{i_1}, x_{i_2}, \cdots, x_{i_s}\}$, then $x_{i_j} \in P_j$, where $j \in \{1, 2, \cdots, s\}$; otherwise, we may as well suppose there exist x_{i_1} and x_{i_2} such that $x_{i_1}, x_{i_2} \in P_1$, then $\beta_{i_1} = \beta_{i_2}$. Thus β_{i_1} and β_{i_2} are linearly dependent in $V(n, GF(2))$, which makes the columns labeled by B are linearly dependent in $V(n, GF(2))$. That contradicts $B \in \mathcal{B}(M([B(R)])$. Let $\mathbf{v} = (v_1, v_2, \cdots, v_n)^T$, where $v_i = 1$ if and only if $x_i \in B$. Then $\theta(\mathbf{v}) = B$. According to the definition of $B(R)$, we know $B(R)\mathbf{v} = \mathbf{1}$. Thus $\mathcal{B}(M(R)) \subseteq \{\theta(\mathbf{v}) : B(R)\mathbf{v} = \mathbf{1}\}$. Next, we prove the minimality of B. If B is not a minimal set of $\{\theta(\mathbf{v}) : B(R)\mathbf{v} = \mathbf{1}\}$, then there exists $B_1 \in Min\{\theta(\mathbf{v}) - \emptyset : B(R)\mathbf{v} = \mathbf{1}\}$ such that $B_1 \subseteq B$. However, for all $x_{i_j} \in B$, $B - \{x_{i_j}\} \notin \{\theta(\mathbf{v}) : B(R)\mathbf{v} = \mathbf{1}\}$ over $GF(2)$ which implies contradictory. Hence, we have $\mathcal{B}(M(R)) = Min(\{\theta(\mathbf{v}) : B(R)\mathbf{v} = \mathbf{1}\})$.

As we know, for a F-representable matroid, any independent set I is defined by the linear independence of columns, labeled by I, of a F-representation for the matroid. The following proposition establishes another representation of the independent set over binary field.

Proposition 6. *Let R be an equivalence relation on U and $B(R)$ a matrix representation of R over $GF(2)$. $\mathcal{I}(M(R)) = Low(Min(\{\theta(\mathbf{v}) : B(R)\mathbf{v} = \mathbf{1}\}))$.*

In fact, the space proposed in above proposition has closely relationship to rough sets.

Proposition 7. *Let R be an equivalence relation on U and $B(R)$ a matrix representation of R over $GF(2)$. $\{\theta(\mathbf{v}) : B(R)\mathbf{v} = \mathbf{1}\} \subseteq \{X : R^*(X) = U\}$.*

Proof. Suppose $U = \{x_1, x_2, \cdots, x_n\}$ and $U/R = \{P_1, P_2, \cdots, P_s\}$ and $B(R) = [\beta_1, \beta_2, \cdots, \beta_n]$ which the columns are labeled, in order, by $x_1, x_2, \cdots, x_n$. For all $X \in \{\theta(\mathbf{v}) : B(R)\mathbf{v} = \mathbf{1}\}$, we may as well suppose $X = \{x_{i_1}, x_{i_2}, \cdots, x_{i_t}\}$ and $\mathbf{v} \in V(n, GF(2))$ such that $\theta(\mathbf{v}) = X$. Then we can obtain $\beta_{i_1} + \beta_{i_2} + \cdots + \beta_{i_t} = \mathbf{1}$ which implies that for all $j \in \{1, 2, \cdots, s\}$, there are odd number of vectors of $\{\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_t}\}$ such that the jth of it has non-zero element. According to the definition of $B(R)$, then $X \bigcap P_i \neq \emptyset$ for all $i \in \{1, 2, \cdots, s\}$, that is, $R^*(X) = U$. Thus $X \subseteq \{X : R^*(X) = U\}$, that is, $\{\theta(\mathbf{v}) : B(R)\mathbf{v}) = \mathbf{1}\} \subseteq \{X : R^*(X) = U\}$.

However, for all $X \subseteq U$, $\theta(X)$ may not satisfy $B(R)\theta(X) = \mathbf{1}$ over binary field.

Example 5. Let us revisit Example 2. Suppose $X = \{1, 2, 3\}$. Then $\mathbf{v} = (1, 1, 1, 0, 0)^T$ and $\theta(\mathbf{v}) = X$. It is clear that $R^*(X) = U$, but $B(R)\mathbf{v} = \beta_1 + \beta_2 + \beta_3 = (0, 1)^T$. Hence $\{X : R^*(X) = U\} \nsubseteq \{\theta(\mathbf{v}) : B(R)\mathbf{v}) = \mathbf{1}\}$.

The following proposition connects the base of the matroid induced by an equivalence relation with a subset of universe of discourse which the approximation of the set is equal to the universe of discourse.

Proposition 8. *Let R be an equivalence relation on U. $\mathcal{B}(M(R)) = Min(\{X : R^*(X) = U\})$.*

Proof. Suppose $U = \{x_1, x_2, \cdots, x_n\}$ and $U/R = \{P_1, P_2, \cdots, P_s\}(s \leq n)$. According to Theorem 3 and Proposition 7, we know $\mathcal{B}(M(R)) \subseteq Min(\{X \subseteq U : R^*(X) = U\})$. For all $B \in Min(\{X \subseteq U : R^*(X) = U\})$, we know $B = \{b_1, b_2, \cdots, b_s\}$ and $b_i \in P_i$ for all $i \in \{1, 2, \cdots, s\}$ according to the minimality of B. Similar to the proof of Theorem 3, we know $B \in Min(\{\theta(\mathbf{v}) : B(R)\mathbf{v} = \mathbf{1}\})$, that is, $B \in \mathcal{B}(M(R))$. Hence we prove the result.

The proposition below presents another form of the independent set of the matroid induced by an equivalence relation.

Proposition 9. *Let R be an equivalence relation on U. $\mathcal{I}(M(R)) = Low(Min(\{X : R^*(X) = U\}))$.*

The equivalent characterization of the independent set of the matroid induced by equivalence relation lay the sound foundation for us to study rough sets by matrix approaches.

4 Rough sets constructed by matrix null space over binary field

In this section, we study how to construct an equivalence relation by matrix null space over binary field. It is interesting to find that there is an one-to-one correspondence between the equivalence relations and the minimal null space of matrices which defined in this section. First, we define a relation by the null space of matrix over binary field.

Definition 13. *Let A be an $m \times n$ matrix over F. One can define a relation $R(A)$ on U as follows: for all $x_i, x_j \in U$,*

$$(x_i, x_j) \in R(A) \Leftrightarrow x_i = x_j \text{ or } \mathbf{e}_i + \mathbf{e}_j \in \mathcal{N}(A),$$

where $\mathbf{e}_i, \mathbf{e}_j \in V(n, F)$ *and* $\theta(\mathbf{e}_i) = \{x_i\}$ *and* $\theta(\mathbf{e}_j) = \{x_j\}$.

Example 6. $B(R)$ is shown in Example 2 and $F = GF(2)$, then we know $U = \{1, 2, 3, 4, 5\}$. For $x_i, x_j \in U$, if $x_i = x_j$, then $(x_i, x_j) \in R$, thus $(x_i, x_i) \in R$ for all $x_i \in U$. $\mathbf{e}_1 = (1, 0, 0, 0, 0)^T$ and $\mathbf{e}_3 = (0, 0, 1, 0, 0)^T$, then $\theta(\mathbf{e}_1) = \{x_1\}$ and $\theta(\mathbf{e}_3) = \{x_3\}$, moreover, $B(R)(\mathbf{e}_1 + \mathbf{e}_3) = \mathbf{0}$, that is, $\mathbf{e}_1 + \mathbf{e}_3 \in \mathcal{N}(B(R))$. Hence $(1, 3) \in R(B(R))$. Using the same method, we can obtain $R(B(R)) = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (1, 3), (3, 1), (2, 4), (4, 2), (2, 5), (5, 2), (4, 5), (5, 4)\}$.

The relation defined above may not be an equivalence relation over a field, but it is an equivalence relation over binary field.

Proposition 10. *Let* $A = (a_{ij})_{m \times n} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ *over* $GF(2)$, *where for all* $i \in \{1, 2, \cdots, n\}$, $\alpha_i \neq \mathbf{0}$. $R(A)$ *is an equivalence relation over* $GF(2)$.

Proof. The reflexivity and symmetry of $R(A)$ are obvious. Now we prove the transitivity of $R(M)$. $\forall x_i, x_j, x_k \in U$, there exist identity vectors $\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k \in V(n, GF(2))$ such that $\theta(\mathbf{e}_i) = \{x_i\}$, $\theta(\mathbf{e}_j) = \{x_j\}$ and $\theta(\mathbf{e}_k) = \{x_k\}$. Because $\mathbf{0} = \mathbf{0} - \mathbf{0} = A(\mathbf{e}_i + \mathbf{e}_j) - A(\mathbf{e}_j + \mathbf{e}_k) = A\mathbf{e}_i + A\mathbf{e}_j - A\mathbf{e}_j - A\mathbf{e}_k = A\mathbf{e}_i - A\mathbf{e}_k = A\mathbf{e}_i + A(-\mathbf{e}_k) = A\mathbf{e}_i + A\mathbf{e}_k = A(\mathbf{e}_i + \mathbf{e}_k)$, then $\mathbf{e}_i + \mathbf{e}_k \in \mathcal{N}(A)$. Hence we can obtain the result that if $x_i R x_j$ and $x_j R x_k$, then $x_i R x_j$.

The relation of two elements x, y defined in definition 13 is defined through the linear relativity of the columns, labeled by x and y, respectively, of a given matrix over a field. Hence, it is not difficult for us to obtain the following proposition.

Proposition 11. *Let* A_1 *and* A_2 *be two matrices which do not contain zero columns over* $GF(2)$. *If* $M_{GF(2)}[A_1] = M_{GF(2)}[A_2]$, *then* $R(A_1) = R(A_2)$.

Proof. Suppose $U = \{x_1, x_2, \cdots, x_n\}$ and the columns of $A_1 = (\alpha_1, \alpha_2, \cdots, \alpha_n)$ are, in order, labeled by $x_1, x_2, \cdots, x_n$, so does $A_2 = (\alpha_1^{'}, \alpha_2^{'}, \cdots, \alpha_n^{'})$. $\forall (x_i, x_j) \in R(A_1)$, then $x_i = x_j$ or $\mathbf{e}_i + \mathbf{e}_j \in \mathcal{N}(A_1)$, where $\mathbf{e}_i, \mathbf{e}_j \in V(n, GF(2))$ and $\theta(\mathbf{e}_i) = \{x_i\}$ and $\theta(\mathbf{e}_j) = \{x_j\}$. Thus $\theta(\mathbf{e}_i + \mathbf{e}_j) = \{x_i, x_j\}$. According to Proposition 4, then $\{x_i, x_j\} \in \mathcal{D}(M_{GF(2)}[A_1]) = \mathcal{D}(M_{GF(2)}[A_2])$ for $M_{GF(2)}[A_1] = M_{GF(2)}[A_2]$. Thus the columns of A_2 labeled by x_i and x_j, respectively, are linearly dependent in $V(n, GF(2))$, that is, $\mathbf{0} = \alpha_i^{'} + \alpha_j^{'} = A_2(\mathbf{e}_i + \mathbf{e}_j)$ because A_2 does not contain zero columns. Hence $\mathbf{e}_i + \mathbf{e}_j \in \mathcal{N}(A_2)$, that is, $(x_i, x_j) \in R(A_2)$. That implies $R(A_1) \subseteq R(A_2)$. Similarly, we can obtain $R(A_1) \supseteq R(A_2)$. Thus $R(A_1) = R(A_2)$.

As we know, a F-representation for a F-representable matroid may not be unique, so dose $GF(2)$. But Proposition 11 indicates that the equivalence relations induced by different $GF(2)$-representation for a $GF(2)$-representable matroid are the same. Proposition 1 establishes an approach to induce a matroid from an equivalence relation and we know the matroid is $GF(2)$-representable. Over binary field, what is the relationship between the equivalence relation induced by the null space of a $GF(2)$-representation for the matroid and the original equivalence relation?

Proposition 12. *Let R be an equivalence relation on U. If $M(R) = M_{GF(2)}[A(R)]$, then $R(A(R)) = R$.*

Proof. Let $U = \{x_1, x_2, \cdots, x_n\}$. Since for all $i \in \{1, 2, \cdots, n\}$, $\{x_i\}$ is an independent set of $M(R)$. Thus $A(R)$ dose not contain zero columns. For all $(x_i, x_j) \in R(A(R))$, then $x_i = x_j$ or $\mathbf{e}_i + \mathbf{e}_j \in \mathcal{N}(A(R))$, where $\mathbf{e}_i, \mathbf{e}_j \in V(n, GF(2))$ and $\theta(\mathbf{e}_i) = \{x_i\}$ and $\theta(\mathbf{e}_j) = \{x_j\}$. If $x_i = x_j$, then $(x_i, x_j) \in R$ for R is an equivalence relation. If $x_i \neq x_j$, then $\mathbf{e}_i + \mathbf{e}_j \in \mathcal{N}(A(R))$. According to Proposition 4, then $\theta(\mathbf{e}_i + \mathbf{e}_j) = \{x_i, x_j\} \in \mathcal{D}(M(R))$. But $\{x_i\}$ or $\{x_j\}$ is an independent set of $M(R)$. Thus $\{x_i, x_j\} \in \mathcal{C}(M(R))$, that is, there exists $P \in U/R$ such that $\{x_i, x_j\} \in P$ which implies $(x_i, x_j) \in R$. Conversely, $\forall (x_i, x_j) \in R$, then there exists $P \in U/R$ such that $\{x_i, x_j\} \in P$, that is, $\{x_i, x_j\} \in \mathcal{C}(M(R))$. According to Theorem 2, then $\mathbf{e}_i + \mathbf{e}_j \in \mathcal{N}(A(R))$, that is, $(x_i, x_j) \in R(A(R))$. Thus $R(A(R)) = R$.

Since matrix $B(R)$ is a $GF(2)$-representation for the matroid induced by an equivalence relation, then we can obtain the following corollary.

Corollary 5. *Let R be an equivalence relation and $B(R)$ the matrix representation of R. $R(B(R)) = R$ holds over binary field.*

In fact, there has closely relationship between the equivalence relation and the null space of a matrix. So we first define a special type of matrix.

Definition 14. *Let F be a field and $A = (a_{ij})_{m\times n} = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. If A satisfies the following conditions:*
(1) for all $i \in \{1, 2, \cdots, n\}$, $\alpha_i \neq 0$,
(2) for all $k \in \{2, \cdots, n\}$, if $r_F(\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_k}) < k$, then there exists $\{\alpha_{i_p}, \alpha_{i_q}\} \subseteq \{\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_k}\}$ such that $r_F(\alpha_{i_p}, \alpha_{i_q}) < 2$,
then A is called binary dependence matrix and we denote the set of this matrices as $\mathcal{A}$.

The following proposition establishes that a matrix representation of an equivalence relation, namely, $B(R)$, is a binary dependence matrix.

Proposition 13. *Let R be an equivalence relation on U and $B(R) = (b_{ij})_{s\times n}$ the matrix representation of R. $B(R) \in \mathcal{A}$.*

Proof. Suppose $U = \{x_1, x_2, \cdots, x_n\}$ and $U/R = \{P_1, P_2, \cdots, P_s\}$ and the columns of $B(R) = [\beta_1, \beta_2, \cdots, \beta_n]$ which labeled, in order, by $x_1, x_2, \cdots, x_n$. According to the definition of $B(R)$, we know $\beta_i \neq \mathbf{0}$ for all $i \in \{1, 2, \cdots, n\}$. Since $M(R) = M_F[B(R)]$, for all $k \geq 2$, if $\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_k}$ are linearly dependent over F, then $\{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\} \in D(M(R))$. Thus there exists $C \in C(M(R))$ such that $C \subseteq \{x_{i_1}, x_{i_2}, \cdots, x_{i_k}\}$. According to Proposition 1, we may as well suppose $C = \{x_{i_p}, x_{i_q}\}$ and there exists $P_k \in U/R$ such that $x_{i_p}, x_{i_q} \in P_k$. According to the definition of $B(R)$, we know $\beta_{i_p} = \beta_{i_q}$, that is, β_{i_p}, β_{i_q} are linearly dependent over F .

If we first convert a matrix into an equivalence relation, then covert the equivalence relation into a matrix, the minimal null space of the second conversion is the reverse of the minimal null space of the first conversion. In other word, for a matrix $A \in \mathcal{A}$, we can induce an equivalence relation $R(A)$ from A using the method

of Definition 13 over binary field, and then induce a matroid $M(R(A))$ from $R(A)$ through the method of Proposition 1. Since the matroid is $GF(2)$-representable, then $M(R(A)) = M_{GF(2)}[A(R(A))]$, that is, we obtain the other matrix $A(R(A))$. And the minimal null space of A and the one of $A(R(A))$ are the same. First, we can obtain the following proposition.

Proposition 14. *Let $A \in \mathcal{A}$ and $GF(2)$ a field.* $M_{GF(2)}[A(R(A))] = M_{GF(2)}[A]$.

Proof. Suppose $U = \{x_1, x_2, \cdots, x_n\}$ and the columns of A are labeled, in order, by $x_1, x_2, \cdots, x_n$, so dose $A(R(A))$. For all $C \in \mathcal{C}(M_{GF(2)}[A(R(A))])$, according to Proposition 1, we may suppose $C = \{x_i, x_j\}$ and there exists $P_i \in U/R(A)$ such that $\{x_i, x_j\} \in P_i$, that is, $(x_i, x_j) \in R(A)$. Then for $\mathbf{e}_i, \mathbf{e}_j \in V(n, GF(2))$ such that $\theta(\mathbf{e}_i) = \{x_i\}$ and $\theta(\mathbf{e}_j) = \{x_j\}$, we have $A(\mathbf{e}_i + \mathbf{e}_j) = 0$, that is, the columns of A labeled by x_i and x_j are linearly dependent in $V(n, GF(2))$. Since $A \in \mathcal{A}$, A dose not contain zero columns, thus the column of A labeled by x_i or x_j is linearly independent in $V(n, GF(2))$. Hence $C \in \mathcal{C}(M_{GF(2)}[A])$, that is, $\mathcal{C}(M_{GF(2)}[A(R(A))]) \subseteq \mathcal{C}(M_{GF(2)}[A])$. Conversely, $\forall C \in \mathcal{C}(M_{GF(2)}[A])$, the columns of A labeled by the elements of C are linearly dependent in $V(n, GF(2))$. Since $A \in \mathcal{A}$, then there exists C_1 which has only two elements such that $C_1 \subseteq C$. We may as well suppose $C_1 = \{x_i, x_j\}$. Since $A \in \mathcal{A}$, the column of A labeled by x_i or x_j is linearly independent in $V(n, GF(2))$. Thus $C_1 \in \mathcal{C}(M_{GF(2)}[A])$. Based on circuit axiom and Theorem 2, we can obtain $C = C_1 = \{x_1, x_j\} \in Min\{\theta(\mathcal{N}(A)) - \emptyset\}$, that is, $\mathbf{e}_i + \mathbf{e}_j \in \mathcal{N}(A)$. Hence, $(x_i, x_j) \in R(A)$, that is, $C = \{x_i, x_j\} \in \mathcal{C}(M(R(A))) = \mathcal{C}(M_{GF(2)}[A(R(A))])$. Therefore, we obtain $\mathcal{C}(M_{GF(2)}[A]) \subseteq \mathcal{C}(M_{GF(2)}[A(R(A))])$, that is, $M_{GF(2)}[A(R(A))] = M_{GF(2)}[A]$.

The following result is the combination of Theorem 2 and Proposition 14.

Proposition 15. *Let $A \in \mathcal{A}$ and $GF(2)$ a field.*

$$Min\{\theta(\mathcal{N}(A(R(A)))) - \emptyset\} = Min\{\theta(\mathcal{N}(A)) - \emptyset\}$$

holds over $GF(2)$.

Next, we define a operator from the set of equivalence relations to the set of minimal null spaces of the binary dependence matrices.

Definition 15. *An operator $f : \mathbb{R} \to Min\{\theta(\mathcal{N}(\mathcal{A})) - \emptyset\}$ is defined as follows:*

$$f(R) = Min\{\theta(\mathcal{N}(A(R)) - \emptyset\}.$$

Since there is an one-to-one correspondence between the equivalence relations and the 2-matroids, we can obtain the following proposition.

Proposition 16. *Let $GF(2)$ be a field. f is a bijection over $GF(2)$*

Proof. Since an equivalence relation is the only one that determines a matroid, then for all $R_1, R_2 \in \mathbb{R}$ and $R_1 \neq R_2$, $M(R_1) = M_{GF(2)}[A(R_1)] \neq M_{GF(2)}[A(R_2)] = M(R_2)$, that is, $\mathcal{C}(M(R_1)) \neq \mathcal{C}(M(R_2)$. According to Theorem 2, we have the result $Min\{\theta(\mathcal{N}(A(R_1)) - \emptyset\} \neq Min\{\theta(\mathcal{N}(A(R_2)) - \emptyset\}$. Hence f is an injection. For any $Min\{\theta(\mathcal{N}(A) - \emptyset\}$, let $R = R(A)$, then $f(R(A)) = Min\{\theta(\mathcal{N}(A) - \emptyset\}$, that is, f is a surjection. Therefore, f is a bijection.

So we construct an one-to-one correspondence between the equivalence relations and the minimal null spaces of matrices which defined in Definition 14. Moreover, we know there exists an one-to-one correspondence between the equivalence relations and 2-circuit matroids. In fact, we also construct an one-to-one correspondence between the 2-circuit matroids and the minimal null spaces of the binary dependence matrixes used the way of Theorem 2.

5 Conclusions

In this paper, we employed the matrix approaches, namely, null space to study rough sets over a field F. The circuits of a F-representable matroid induced by equivalence relation have closely relationship to the null space of a F-representation for the matroid. We also constructed an one-to-one correspondence between the equivalence relations and minimal null spaces of the matrixes defined in the paper. Though some works have been studied in this paper, there are also many interesting topics deserving further investigation. First, the special properties of a matroid induced by an equivalence relation over another field, for example, ternary field. Second, the representability of a matroid induced by covering or any binary relation. Nullity has closely relationship with the null space, so we also can study rough sets from this viewpoint.

6 Acknowledgments

This work is supported in part by the National Natural Science Foundation of China under Grant No. 61170128, the Natural Science Foundation of Fujian Province, China, under Grant Nos. 2011J01374 and 2012J01294, and the Science and Technology Key Project of Fujian Province, China, under Grant No. 2012H0043.

References

1. Baesens, B., Setiono, R., Mues, C., Vanthienen, J.: Using neural network rule extraction and decision tables for credit-risk evaluation. Management Science March **49** (2003) 312–329
2. Barnabei, M., Nicoletti, G., Pezzoli, L.: Matroids on partially ordered sets. Advances in Applied Mathematics **21** (1998) 78–112
3. Chen, D., Hu, Q., Yang, Y.: Parameterized attribute reduction with gaussian kernel based fuzzy rough sets. Information Sciences **181** (2011) 5169–5179
4. Chen, Y., Miao, D., Wang, R., Wu, K.: A rough set approach to feature selection based on power set tree. Knowledge-Based Systems **24** (2011) 275–281
5. C.Lay, D., ed.: Linear Algebra and Its Applycations. Publishing House of Electronics Industry (2010)
6. Cruz-Cano, R., Lee, M.L.T., Leung, M.Y.: Logic minimization and rule extraction for identification of functional sites in molecular sequences. BioData Mining **5** (2012)
7. Dash, M., Liu, H.: Consistency-based search in feature selection. Artificial Intelligence **151** (2003) 155–176
8. Du, Y., Hu, Q., Zhu, P., Ma, P.: Rule learning for classification based on neighborhood covering reduction. Information Sciences **181** (2011) 5457–5467

9. Edmonds, J.: Matroids and the greedy algorithm. Mathematical Programming **1** (1971) 127–136
10. He, Q., Wu, C., Chen, D., Zhao, S.: Fuzzy rough set based attribute reduction for information systems with fuzzy decisions. Knowledge-Based Systems **24** (2011) 689–696
11. Hu, Q., Yu, D., Liu, J., Wu, C.: Neighborhood rough set based heterogeneous feature subset selection. Information Sciences **178** (2008) 3577–3594
12. Huang, A., Zhu, W.: On matrix representation of three types of covering-based rough sets. In: Granular Computing. (2012) 215–220
13. Min, F., Zhu, W.: Attribute reduction of data with error ranges and test costs. Information Sciences (doi: 10.1016/j.ins.2012.04.031) (2012)
14. Lai, H.: Matroid theory. Higher Education Press, Beijing (2001)
15. Li, X., Liu, S.: Matroidal approaches to rough set theory via closure operators. International Journal of Approximate Reasoning **53** (2012) 513–527
16. Liu, G.: Closures and topological closures in quasi-discrete closure. Applied Mathematics Letters **23** (2010) 772–776
17. Liu, G.: The transitive closures of matrices over distributive lattices. In: Granular Computing. (2006) 63–66
18. Liu, G.: Rough set theory based on two universal sets and its applications. Knowledge-Based Systems **23** (2010) 110–115
19. Liu, Y., Zhu, W.: Characteristic of partition-circuit matroid through approximation number. In: Granular Computing. (2012) 376–381
20. Min, F., He, H., Qian, Y., Zhu, W.: Test-cost-sensitive attribute reduction. Information Sciences **181** (2011) 4928–4942
21. Oxley, J.G.: Matroid theory. Oxford University Press, New York (1993)
22. Pawlak, Z.: Rough sets. International Journal of Computer and Information Sciences **11** (1982) 341–356
23. Tseng, T.L.B., Huang, C.C.: Rough set-based approach to feature selection in customer relationship management. Omega **35** (2007) 365–383
24. Wang, X., Tsang, E.C., Zhao, S., Chen, D., Yeung, D.S.: Learning fuzzy rules from fuzzy samples based on rough set technique. Information Sciences **177** (2007) 4493–4514
25. Wang, S., Zhu, W.: Matroidal structure of covering-based rough sets through the upper approximation number. International Journal of Granular Computing, Rough Sets and Intelligent Systems **2** (2011) 141–148
26. Wang, S., Zhu, Q., Zhu, W., Min, F.: Matroidal structure of rough sets and its characterization to attribute reduction. Knowledge-Based Systems (http://dx.doi.org/10.1016/j.knosys.2012.06.006) (2012)
27. Xiong, Q., ed.: Modern Algebra. Wuhan University Press (1984)
28. Yao, Y., Zhao, Y.: Attribute reduction in decision-theoretic rough set models. Information Sciences **178** (2008) 3356–3373
29. Zhu, W., Wang, S.: Matroidal approaches to generalized rough sets based on relations. International Journal of Machine Learning and Cybernetics **2** (2011) 273–279